Scattered Interpolation Survey

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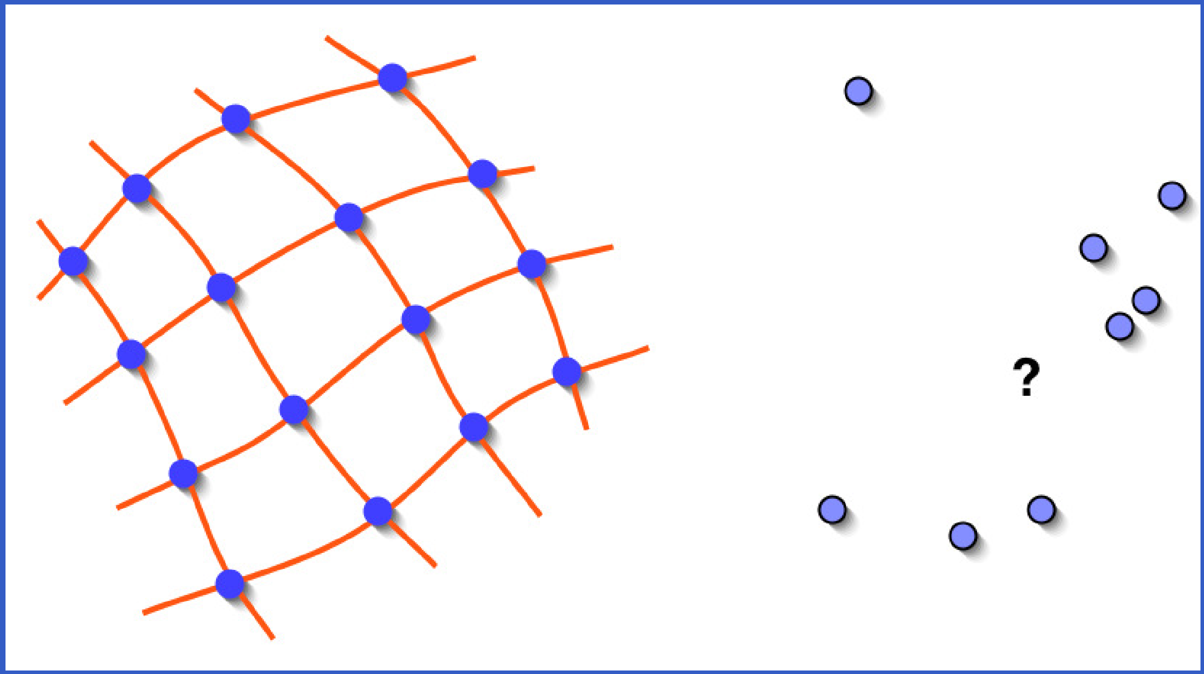
Scattered vs. Regular domain



Motivation

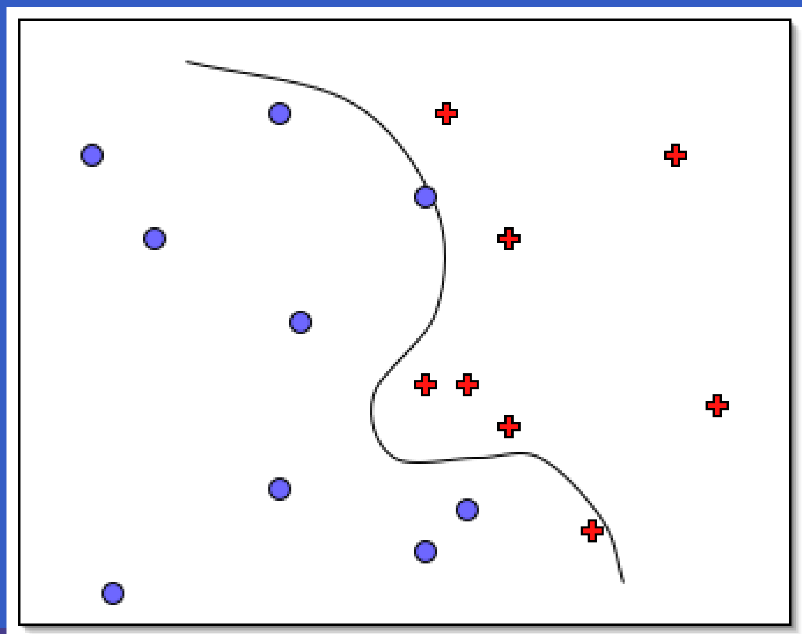
- modeling
- animated character deformation
- texture synthesis
- stock market prediction
- neural networks
- machine learning...

Machine Learning

- Score credit card applicants
- Each person has N attributes: income, age, gender, credit rating, zip code, ...
- i.e. each person is a point in an N-dimensional space
- training data: some individuals have a score “1” = grant card, others “-1” = deny card

Machine Learning

- From training data, learn a function $R^N \rightarrow -1, 1$
- .... by interpolating the training data



Texture Synthesis

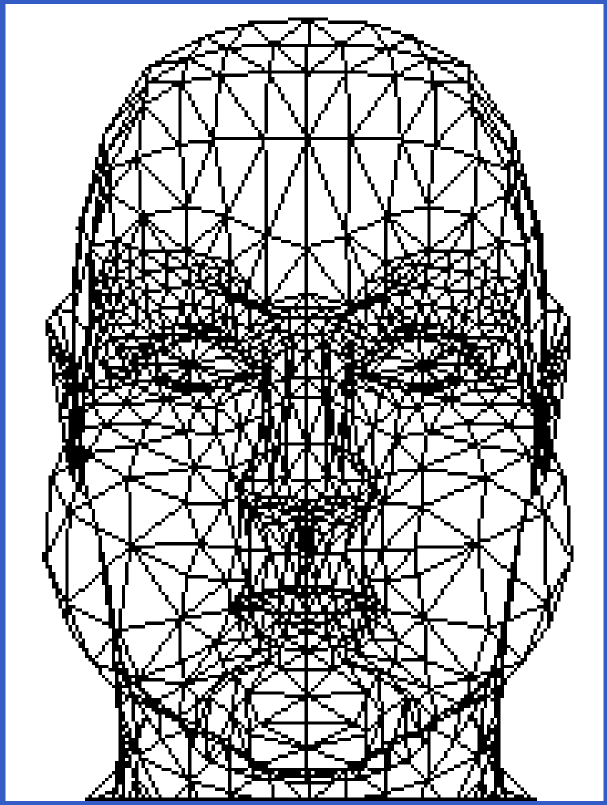
(blackboard drawing)

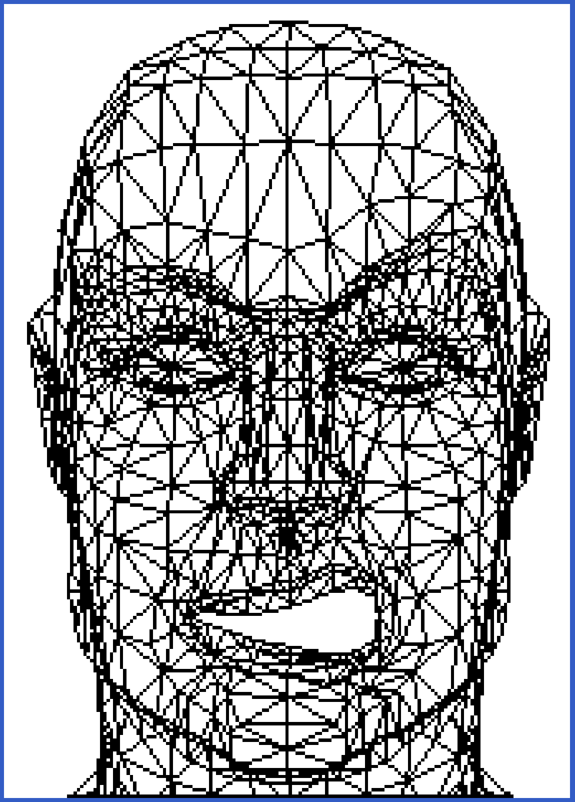
Stock Market Prediction

(blackboard drawing)

Modeling

Deforming a face mesh





Images: Jun-Yong Noh and Ulrich Neumann, CGIT lab

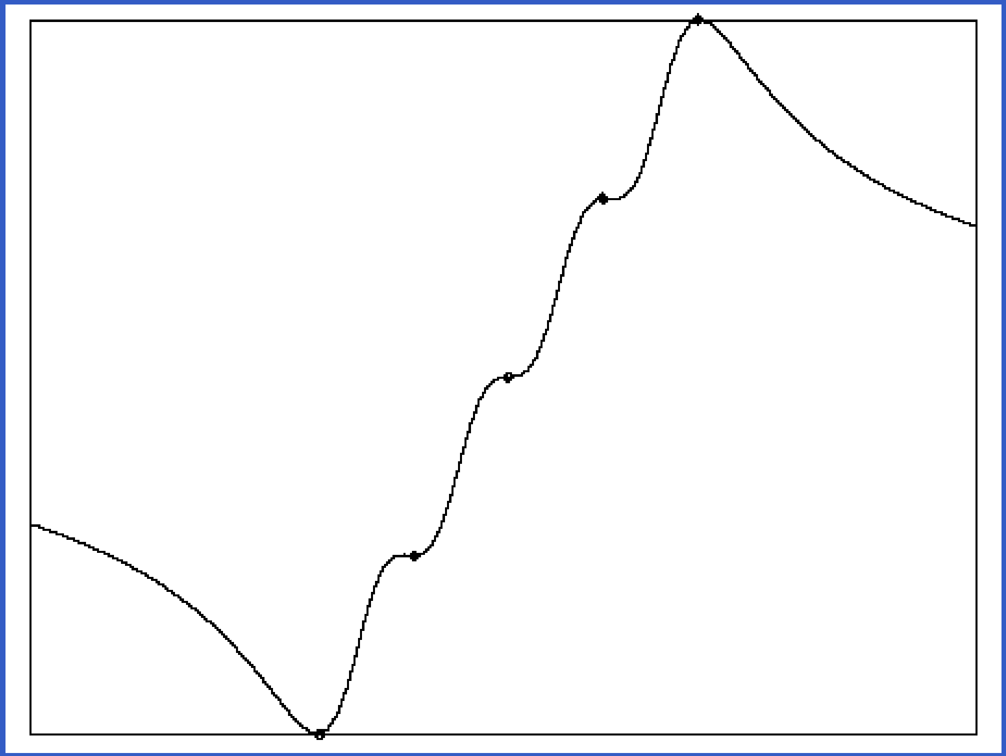
Shepard Interpolation

$$\hat{d}(\mathbf{x}) = \frac{\sum w_k(\mathbf{x}) d_k}{\sum w_k(\mathbf{x})}$$

weights set to an inverse power of the distance: $w_k(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_k\|^{-p}$.

Note: singular at the data points $\mathbf{x}_k$.

Shepard Interpolation



improved “higher order” versions in Lancaster Curve and Surface Fitting book

Natural Neighbor Interpolation

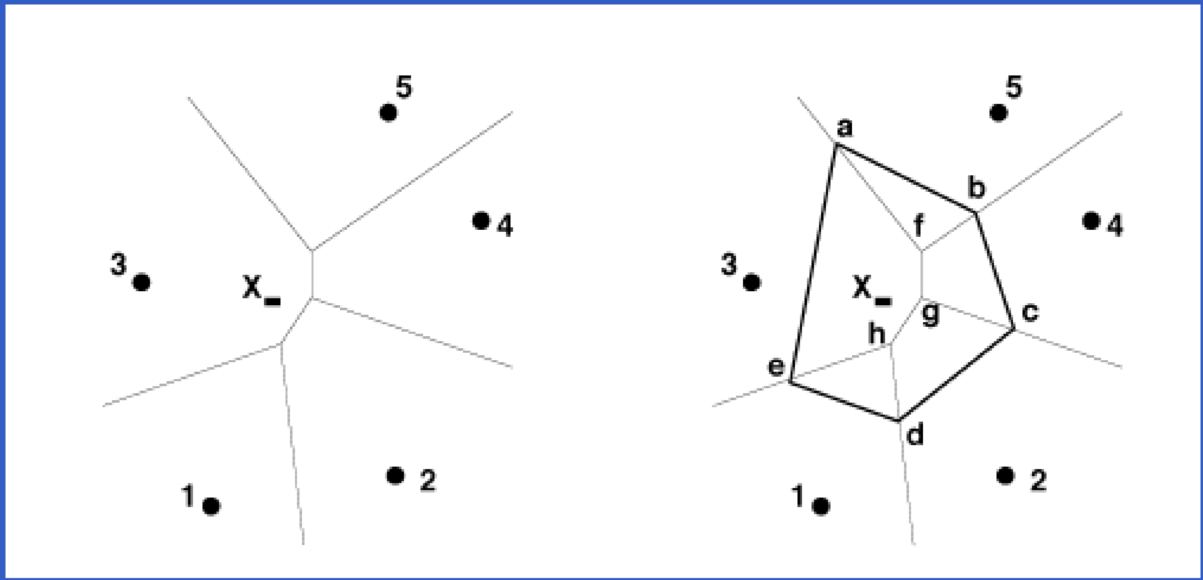


Image: N. Sukmar, Natural Neighbor Interpolation and the Natural Element Method (NEM)

Wiener interpolation

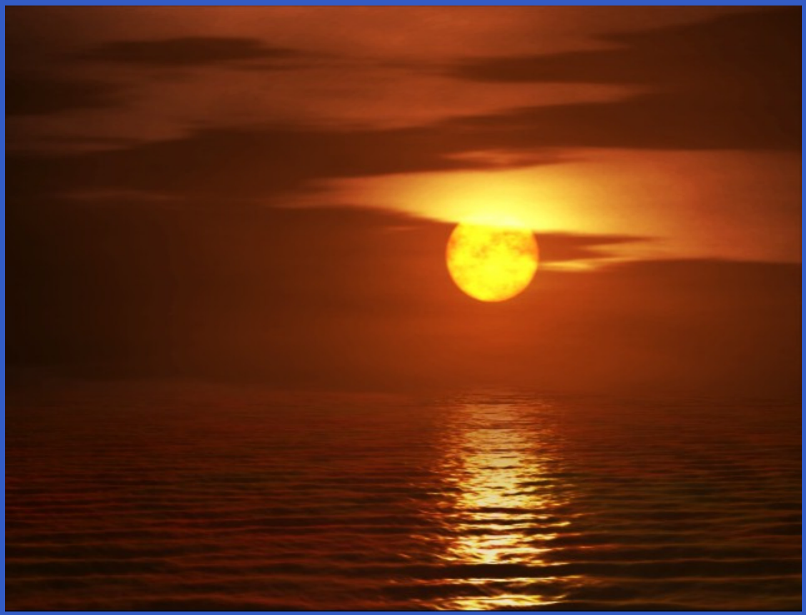
linear estimator	$\hat{x}_t = \sum w_k x_{t+k}$
orthogonality	$E[(x_t - \hat{x}_t)x_m] = 0$
	$E[x_t x_m] = E[\sum w_k x_{t+k}\, x_m]$
autocovariance	$E[x_t x_m] = R(t-m)$
linear system	$R(t-m) = \sum w_k R(t+k-m)$

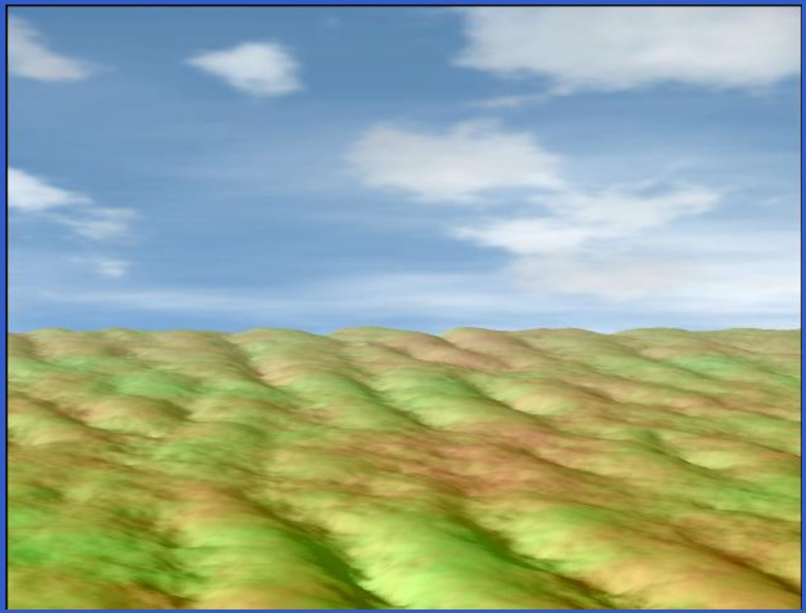
Note no requirement on the actual spacing of the data.

Related to the “Kriging” method in geology.

Applications: Wiener interpolation

Lewis, Generalized Stochastic Subdivision, ACM TOG July 1987





Laplace/Poisson Interpolation

Objective: Minimize a roughness measure, the integrated derivative (or gradient) squared:

$$\int \frac{df}{dx}^2 dx$$

$$\int \int |\nabla f|^2 ds$$

Laplace/Poisson interpolation

minimize $\int (f'(x))^2 dx$

should come out like $\frac{d^2 f}{dx^2} = \nabla^2 = 0$

$F(y, y', x) = y'^2$

$\delta F = \frac{\partial F}{dy'} \frac{dy'}{d\epsilon} \delta\epsilon$

$= \frac{\partial F}{dy'} q' \delta\epsilon$

$\frac{\partial F}{dy'} = 2y' = 2\frac{df}{dx}$

$\frac{dE}{d\epsilon} = \int \frac{\partial F}{dy'} q' dx$

now change q' to q

$\int \frac{\partial F}{dy'} q' dx = \frac{\partial F}{dy'} q - \int \frac{d}{dx} \frac{\partial F}{dy'} q dx$

integration by parts

$\frac{\partial F}{dy'} q \Big|_a^b = 0$

because q is zero at both ends

$\frac{dE}{d\epsilon} = -\int \frac{d}{dx} \frac{\partial F}{dy'} q dx = 0$

variation of functional is zero at minimum

$= -2\frac{d}{dx}\frac{df}{dx} = -2\frac{d^2 f}{dx^2} = -2\nabla^2 f = 0$

Laplace/Poisson: Discrete

Local viewpoint:

roughness $$R = \int |\nabla u|^2 du \approx \sum (u_{k+1} - u_k)^2$$

for a particular k: $$\frac{dR}{du_k} = \frac{d}{du_k}[(u_k - u_{k-1})^2 + (u_{k+1} - u_k)^2]$$

$$= 2(u_k - u_{k-1}) - 2(u_{k+1} - u_k) = 0$$

$$u_{k+1} - 2u_k + u_{k-1} = 0 \rightarrow \nabla^2 u = 0$$

Notice: $D^T D = \ldots 1, -2, 1$

Laplace/Poisson Interpolation

Discrete/matrix viewpoint: Encode derivative operator in a matrix D

$$D = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & & \cdots \end{bmatrix}$$

$$\min_f f^T D^T D f$$

Laplace/Poisson Interpolation

$$\min_f f^T D^T D f$$

$$2 D^T D f = 0$$

i.e.

$$\frac{d^2 f}{dx^2} = 0 \;\; \text{or} \;\; \nabla^2 = 0$$

$f = 0$ is a solution; last eigenvalue is zero, corresponds to a constant solution.

Laplace/Poisson: solution approaches

- direct matrix inverse (better: Choleski)
- Jacobi (because matrix is quite sparse)
- Jacobi variants (SOR)
- Multigrid

Jacobi iteration

matrix viewpoint

$$
\begin{aligned}
Ax &= b \\
(D+E)x &= b && \text{split into diagonal D, non-diagonal E} \\
Dx &= -Ex + b \\
x &= -D^{-1}Ex + D^{-1}b && \text{call } B = D^{-1}E, z = D^{-1}b \\
x &\leftarrow Bx + z && D^{-1} \text{ is easy}
\end{aligned}
$$

hope that largest eigenvalue of B is less than 1

Jacobi iteration

Local viewpoint

Jacobi iteration sets each f_k to the solution of its row of the matrix equation, independent of all other rows:

$$\sum A_{rc} f_c = b_r$$

$$\rightarrow \qquad A_{rk} f_k = b_k - \sum_{j \neq k} A_{rj} f_j$$

$$f_k \leftarrow \frac{b_k}{A_{kk}} - \sum_{j \neq k} A_{kj} / A_{kk} f_j$$

Jacobi iteration

apply to Laplace eqn

Jacobi iteration sets each f_k to the solution of its row of the matrix equation, independent of all other rows:

$$\ldots f_{t-1} - 2f_t + f_{t+1} = 0$$

$$2f_t = f_{t-1} + f_{t+1}$$

$$f_k \leftarrow 0.5 * (f[k-1] + f[k+1])$$

In 2D,

```
f[y][x] = 0.25 * ( f[y+1][x] + f[y-1][x] +
f[y][x-1] + f[y][x+1] )
```

But now let's interpolate

1D case, say f_3 is known. Three eqns involve f_3. Subtract (a multiple of) f_3 from both sides of these equations:

$$
\begin{array}{rcl}
f_1 - 2f_2 + f_3 = 0 & \rightarrow & f_1 - 2f_2 + 0 = -f_3 \\
f_2 - 2f_3 + f_4 = 0 & \rightarrow & f_2 + 0 + f_4 = 2f_3 \\
f_3 - 2f_4 + f_5 = 0 & \rightarrow & 0 - 2f_4 + f_5 = -f_3
\end{array}
$$

$$
L = \begin{bmatrix} 1 & -2 & 0 & \\ & 1 & 0 & 1 \\ & & 0 & -2 \\ & & \ldots & \end{bmatrix}
$$

one column is zeroed

Multigrid

r is known, e is not

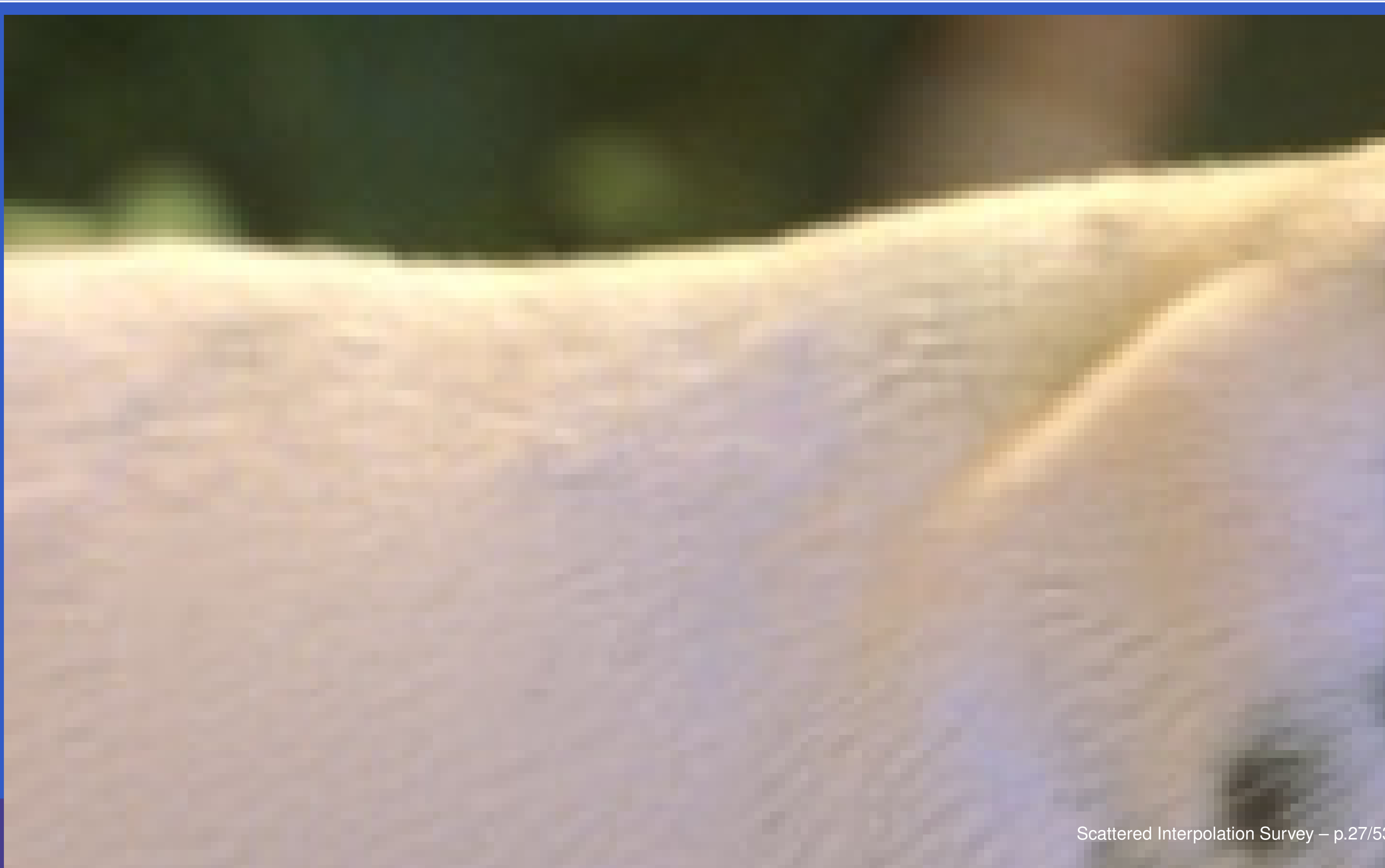
$$
\begin{aligned}
Ax &= b \\
\tilde{x} &= x + e \\
r &= A\tilde{x} - b \\
r &= Ax + Ae - b \\
r &= Ae
\end{aligned}
$$

For Laplace/Poisson, r is smooth. So decimate, solve for e, interpolate. And recurse...

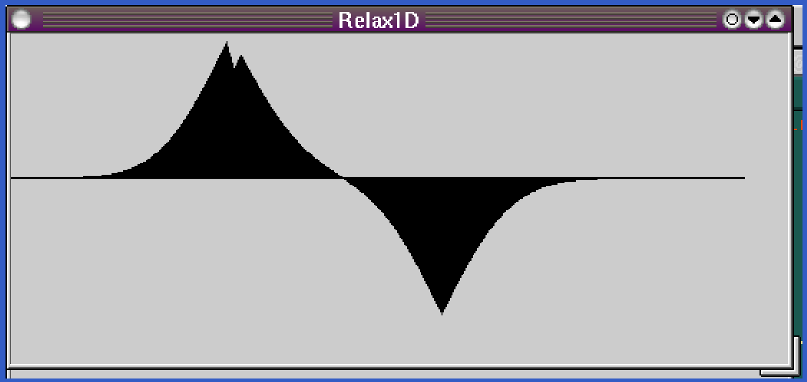
Exciting demo

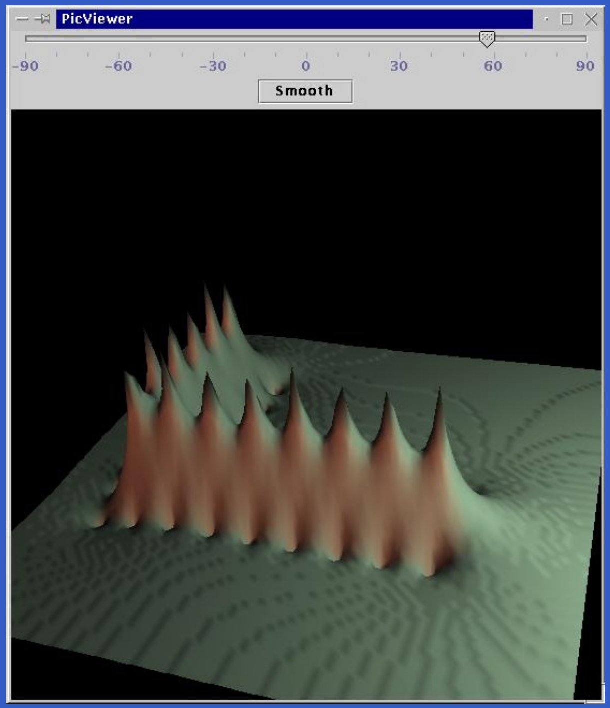
Recovered fur

Recovered fur: detail

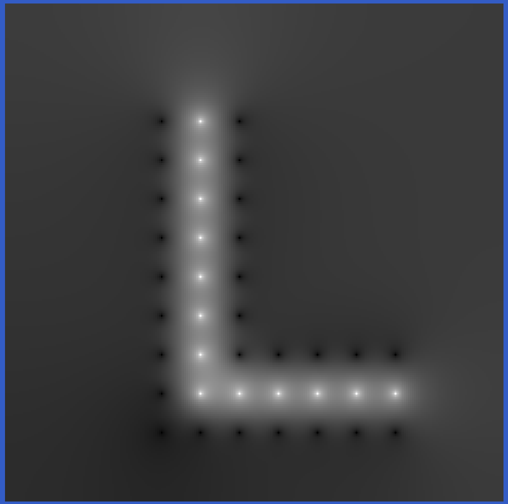


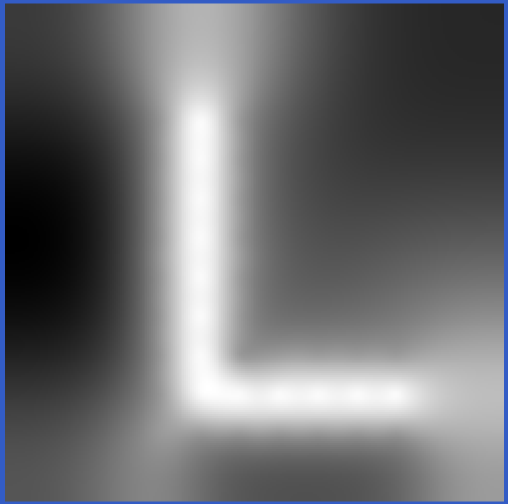
Poor interpolation





Membrane vs. Thin Plate





Left - membrane interpolation, right - thin plate.

Thin plate spline

Minimize the integrated second derivative squared (approximate curvature)

$$\min_f \int \left(\frac{d^2 f}{dx^2} \right)^2 dx$$

Radial Basis Functions

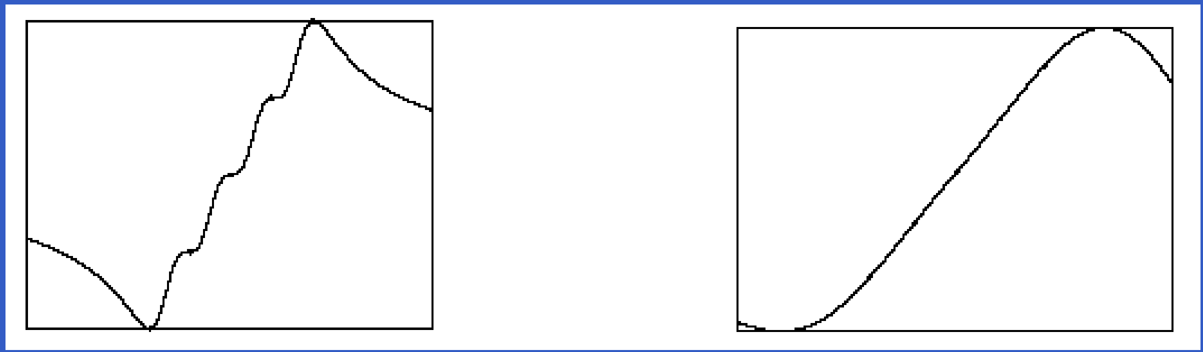
$$\hat{d}(\mathbf{x}) = \sum_{k}^{N} w_k \phi(\|\mathbf{x} - \mathbf{x}_k\|)$$

Radial Basis Functions (RBFs)

- *any* function other than constant can be used!
- common choices:
  - Gaussian $\phi(r) = \exp(-r^2/\sigma^2)$
  - Thin plate spline $\phi(r) = r^2 \log r$
  - Hardy multiquadratic $\phi(r) = \sqrt{(r^2 + c^2)}, c > 0$

Notice: the last two *increase* as a function of radius

RBF versus Shepard's



Solving Thin plate interpolation

- if few known points: use RBF
- if many points use multigrid instead

- but Carr/Beatson et. al. (SIGGRAPH 01) use Greengart FMM for RBF with large numbers of points

Radial Basis Functions

$$\hat{d}(\mathbf{x}) = \sum_{k}^{N} w_k \phi(\|\mathbf{x} - \mathbf{x}_k\|)$$

$$\begin{aligned}
e &= \sum_j (d(\mathbf{x}_j) - \hat{d}(\mathbf{x}_j))^2 \\
&= \sum_j (d(\mathbf{x}_j) - \sum_k^N w_k \phi(\|\mathbf{x}_j - \mathbf{x}_k\|))^2 \\
&= (d(\mathbf{x}_1) - \sum_k^N w_k \phi(\|\mathbf{x}_1 - \mathbf{x}_k\|))^2 + (d(\mathbf{x}_2) - \sum_k^N w_k \phi(\|\mathbf{x}_2 - \mathbf{x}_k\|))^2 + \cdots
\end{aligned}$$

Radial Basis Functions

$$\text{define } R_{jk} = \phi(\|\mathbf{x}_j - \mathbf{x}_k\|)$$

$$\begin{aligned}
&= (d(\mathbf{x}_1) - (w_1 R_{11} + w_2 R_{12} + w_3 R_{13} + \cdots))^2 \\
&\quad +(d(\mathbf{x}_2) - (w_1 R_{21} + w_2 R_{22} + w_3 R_{23} + \cdots))^2 + \cdots \\
&\quad +(d(\mathbf{x}_m) - (w_1 R_{m1} + w_2 R_{m2} + w_3 R_{m3} + \cdots))^2 + \cdots \\
\frac{d}{dw_m} &= 2(d(\mathbf{x}_1) - (w_1 R_{11} + w_2 R_{12} + w_3 R_{13} + \cdots))R_{1m} \\
&\quad +2(d(\mathbf{x}_2) - (w_1 R_{21} + w_2 R_{22} + w_3 R_{23} + \cdots))R_{2m} \\
&\quad +\cdots \\
&\quad +2(d(\mathbf{x}_m) - (w_1 R_{m1} + w_2 R_{m2} + w_3 R_{m3} + \cdots)) + \cdots = 0
\end{aligned}$$

put $R_{k1}, R_{k2}, R_{k3}, \cdots$ in row m of matrix.

Radial Basis Functions

$$\hat{d}(\mathbf{x}) = \sum_{k}^{N} w_k \phi(\|\mathbf{x} - \mathbf{x}_k\|)$$

$$e = ||(\mathbf{d} - \mathbf{\Phi}\mathbf{w})||^2$$

$$e = (\mathbf{d} - \mathbf{\Phi}\mathbf{w})^T(\mathbf{d} - \mathbf{\Phi}\mathbf{w})$$

$$\frac{de}{d\mathbf{w}} = 0 = -\mathbf{\Phi}^T(\mathbf{d} - \mathbf{\Phi}\mathbf{w})$$

$$\mathbf{\Phi}^T\mathbf{d} = \mathbf{\Phi}^T\mathbf{\Phi}\mathbf{w}$$

$$\mathbf{w} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{d}$$

Where does TPS kernel come from

Fit an unknown function f to the data y_k, regularized by minimizing a smoothness term.

$$E[f] = \sum (f_k - y_k)^2 + \lambda \int ||Pf||^2$$

e.g. $$||Pf||^2 = \int \left(\frac{d^2 f}{dx^2}\right)^2 dx$$

Variational derivative of E wrt f leads to a differential equation

$$P'Pf(x) = \frac{1}{\lambda} \sum (f(x) - y_k)\delta(x - x_k)$$

Where does TPS kernel come from

Solve linear differential equation by finding Green's function of the differential operator, convolving it with the RHS (works only for a linear operator). Schematically,

$Lf = rhs$	L is the operator P'P,
	rhs is the data fidelity
$f = g \star rhs$	f obtained by convolving g $\star$ rhs
$Lg = \delta$	choosing rhs $= \delta$ gives this eqn

g is the "convolutional inverse" of L.

Where does TPS kernel come from

In summary, the kernel g is the inverse Fourier transform of the reciprocal of the Fourier transform of the “adjoint-squared” smoothing operator P.

Where does TPS kernel come from

Fit an unknown function f to the data y_k, regularized by minimizing a smoothness term.

$$E[f] = \sum (f_k - y_k)^2 + \lambda \int ||Pf||^2$$

$$\text{e.g.} \qquad ||Pf||^2 = \int \left(\frac{d^2 f}{dx^2} \right)^2 dx$$

A similar discrete version.

$$E[f] = (f-y)' S' S (f-y) + \lambda f' P' P f$$

Where does TPS kernel come from

(continued) A similar discrete version.

$$E[f] = (f-y)'S'S(f-y) + \lambda f'P'Pf$$

- To simplify things, here the data points to interpolate are required to be at discrete sample locations in the vector y, so the length of this vector defines a “sample rate” (reasonable).
- S is a “selection matrix” with 1s and 0s on the diagonal (zeros elsewhere). It has 1s corresponding to the locations of data in y. y can be zero (or any other value) where there is no data.
- P is a diagonal-constant matrix that encodes the discrete form of the regularization operator. E.g. to minimize the integrated curvature, rows of P will contain:

$$\begin{bmatrix} -2, 1, 0, 0, \ldots \\ 1, -2, 1, 0, \ldots \\ 0, 1, -2, 1, \ldots \end{bmatrix}$$

Where does TPS kernel come from

Take the derivative of E with respect to the vector f,

$$2S(f-y) + \lambda 2P'Pf = 0$$

$$P'Pf = -\frac{1}{\lambda}S(f-y)$$

Multiply by G, being the inverse of $P'P$:

$$f = GP'Pf = -\frac{1}{\lambda}GS(f-y)$$

So the RBF kernel “comes from” $G = (P'P)^{-1}$.

Where does TPS kernel come from: D

(Discrete version) RBF kernel is $G = (P'P)^{-1}$.

Take SVD

$$P = UDV' \Rightarrow P'P = VD^2V'$$

The inverse of VD^2V' is $VD^{-2}V'$.

- eigenvectors of a circulant matrix are sinusoids,
- and P is diagonal-constant (toeplitz?), or nearly circulant.
- So $VD^{-2}V'$ is approximately the same as taking the Fourier transform and then the reciprocal (remembering that D are the singular values of P not $P'P$)

Matrix regularization

Find w to minimize $(Rw - b)^T(Rw - b)$. If the training points are very close together, the corresponding columns of R are nearly parallel. Difficult to control if points are chosen by a user.

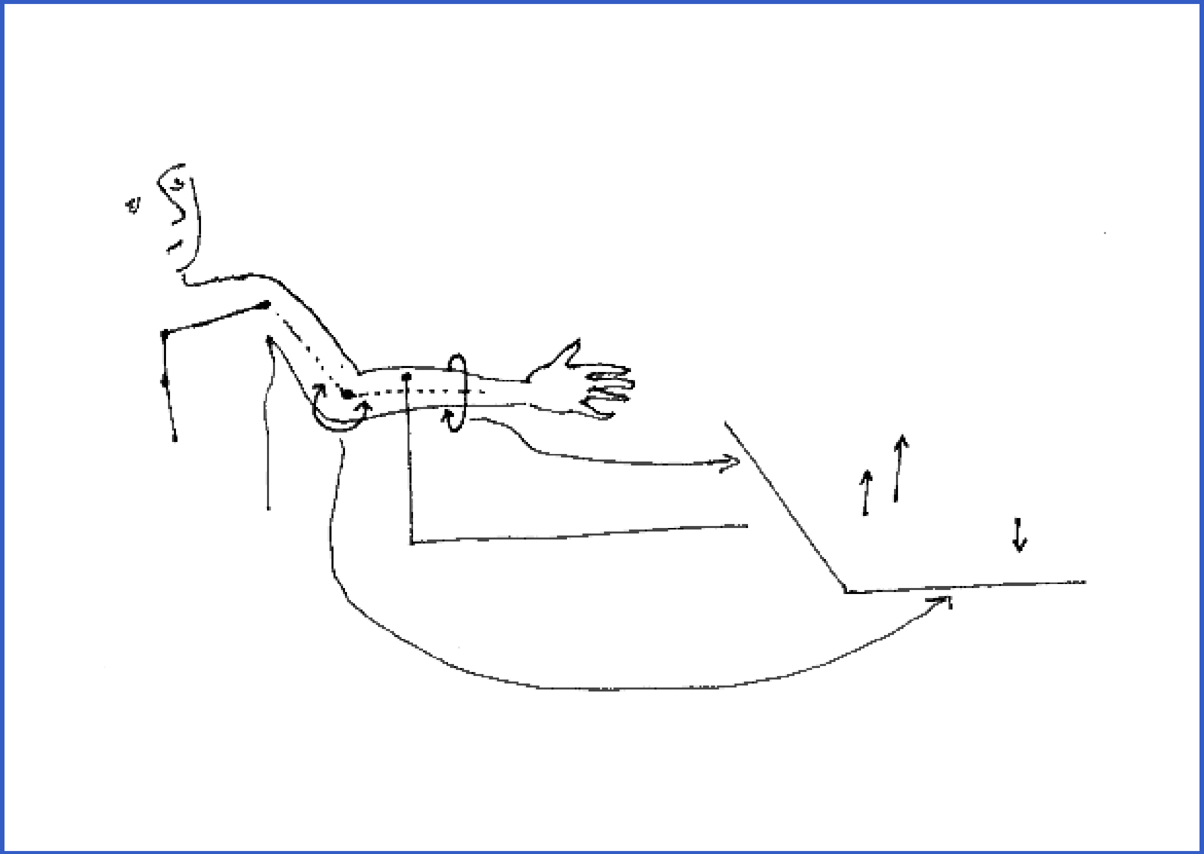
Add a term to keep the weights small: $w^T w$.

$$
\begin{aligned}
\text{minimize} \quad & (Rw - b)^T(Rw - b) + \lambda w^T w \\
& R^T(Rw - b) + 2\lambda w = 0 \\
& R^T Rw + 2\lambda w = R^T b \\
& (R^T R + 2\lambda I)w = R^T b \\
& w = (R^T R + 2\lambda I)^{-1} R^T b
\end{aligned}
$$

Applications: Pose Space Deformation

Lewis/Cordner/Fong, SIGGRAPH 2000

incorporated in Softimage



Applications: Pose Space Deformation

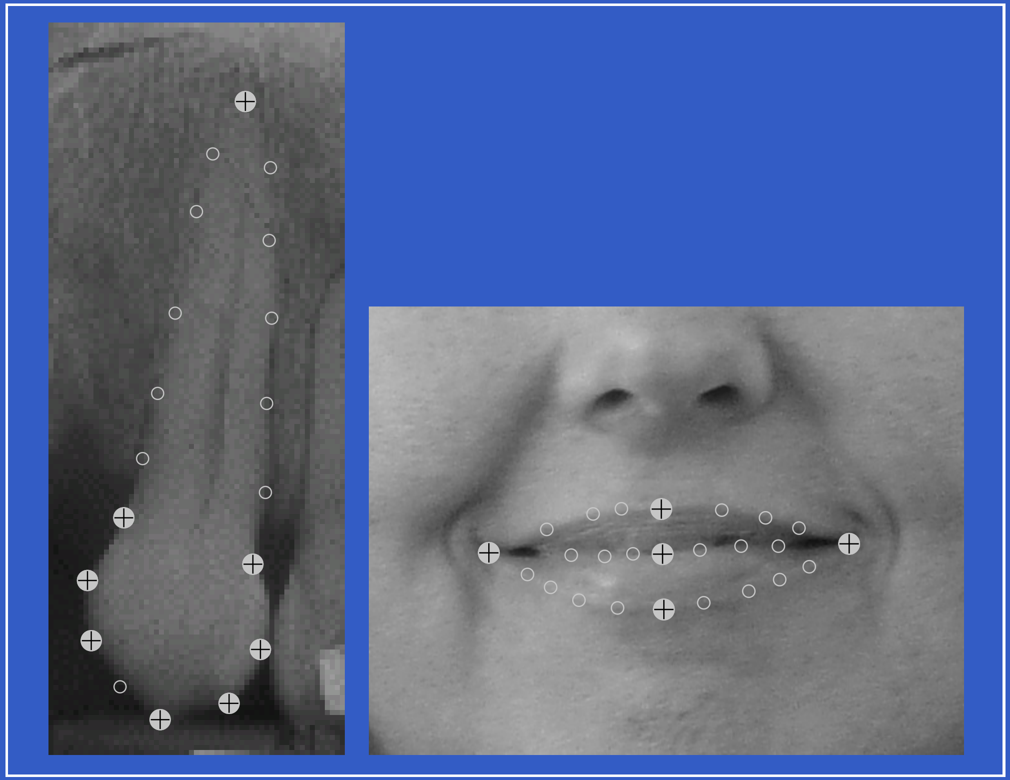
Figure 8a. Comparison of PSD and SSD on an animating shoulder – PSD using only two sculpted poses.

Figure 8b. SSD on an animating shoulder. The shoulder area is especially problematic for SSD due to the large range of rotational movement.

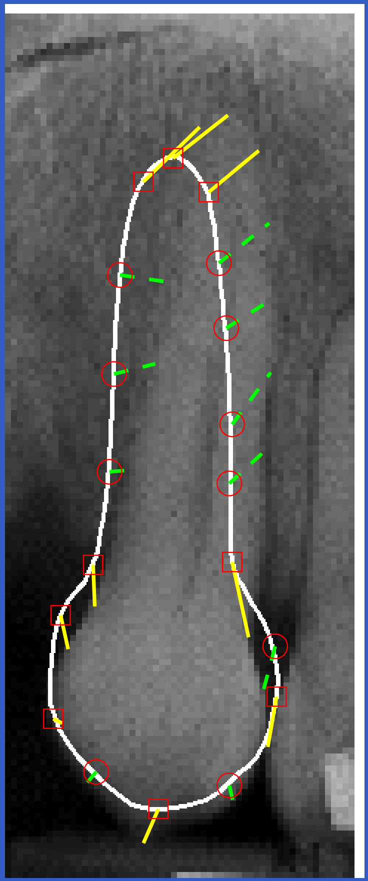
Pose Space Deformation

Applications: Matrix virtual city

Smart Point Placement for Thin Plate



Smart Point Placement for Thin Plate



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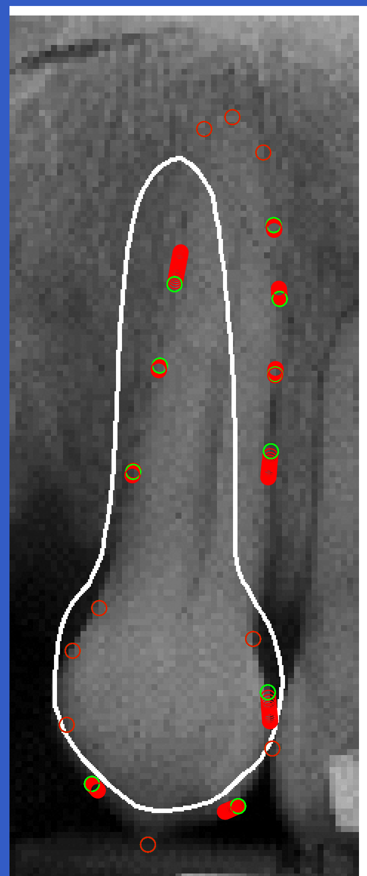

Smart Point Placement for Thin Plate