Understanding Modal analysis (r.e. DyRT paper)

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Some notes (mostly to myself) on understanding how to get the "modal" equations in the DyRT paper.

DyRT Starts with an equation

$$M\ddot{u} + C\dot{u} + Ku = F$$

This is a general second-order linear vector differential equation (u is a vector).

The result of doing the modal transformation is

$$M_q\ddot{q} + C_q\dot{q} + Kq = Q$$

where the matrices M_q, K_q are diagonal.

Two facts from differential equations:

1. the solutions to linear, constant coefficient differential equations are exponentials. These include both exponential decay and increase, $\exp(at)$ and sinusoids (a imaginary).
2. The solution is some combination of the solutions that are found by setting the right-hand-side forcing term (F) to zero (making the "homogeneous" form of the equation).

Starting with the first fact, any solution u will be an exponential $u = e^{\lambda t}v$, where v is a vector. From this, $\dot{u} = \lambda e^{\lambda t}v$ and $\ddot{u} = \lambda^2 e^{\lambda t}v$.

Now apply the second fact (set F to zero), and then substitute $u, \dot{u}, \ddot{u}$ in the homogeneous equation (Ignore the C (friction) part for now):

$$\lambda^2 M e^{\lambda t}v + K e^{\lambda t}v = 0$$

Now divide by $e^{\lambda t}$, giving

$$\lambda^2 Mv + Kv = 0$$

or

$$Kv = -\lambda^2 Mv$$

or

$$M^{-1}Kv = -\lambda^2 v$$

So v is an eigenvector of $M^{-1}K$. In general $M^{-1}K$ will have more than one eigenvector, all of these are possible particular solutions, and the forced solution will be expressable as a linear combination of them.

In the DyRT paper Φ is a matrix whose columns are the eigenvectors v. With this matrix a linear combination of the eigenvectors (i.e. the general solution) is Φq. Substituting this into the origional equation,

$$M\Phi\ddot{q} + C\Phi\dot{q} + K\Phi q = F$$

Now pre-multiply by Φ^T:

$$\Phi^T M\Phi\ddot{q} + \Phi^T C\Phi\dot{q} + \Phi^T K\Phi q = \Phi^T F$$

DyRT relabels $M_q \equiv \Phi^T M\Phi$, etc., giving the second equation at the top.

A mystery: the DyRT paper says that M_q, K_q are diagonal. How can both of M, K be diagonalized by the same Φ matrix? I don't understand that, but it does work out that if Φ diagonalizes either one of M or K then it diagonalizes the other:

$$\Phi^T M^{-1} K \Phi = \Lambda \quad \text{pre and post multiplying by } \Phi \text{ gives a diagonal matrix, with the eigenvalues on the diagonal}$$
$$M^{-1} K \Phi = \Phi\Lambda$$
$$K\Phi = M\Phi\Lambda$$
$$K = M\Phi\Lambda\Phi^T$$
$$\Phi^T K \Phi = ? \quad \text{now given this } K\text{, is it diagonalized by } \Phi\text{?}$$
$$= \Phi^T (M\Phi\Lambda\Phi^T)\Phi$$
$$= \Phi^T M \Phi\Lambda$$

This says that if M is diagonalized by Φ, then K is also.